

On the Pseudocodeword Redundancy

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Abstract—We define the AWGNC, BSC, and max-fractional pseudocodeword redundancy $\rho(\mathcal{C})$ of a code \mathcal{C} as the smallest number of rows in a parity-check matrix such that the corresponding minimum pseudoweight is equal to the minimum Hamming distance of \mathcal{C} . We show that most codes do not have a finite $\rho(\mathcal{C})$. We also provide bounds on the pseudocodeword redundancy for some families of codes, including codes based on designs.

I. INTRODUCTION

Pseudocodewords represent the intrinsic mechanism of failure of binary linear codes under linear-programming (LP) or message-passing (MP) decoding. In [1], the pseudocodeword effective Euclidean weight, or *pseudoweight*, was associated with any pseudocodeword. This concept of pseudoweight was shown to play an analogous role to that of the signal Euclidean distance (AWGNC) or Hamming distance (BSC) in the ML decoding scenario. The minimum pseudoweight of the code \mathcal{C} with respect to a parity-check matrix \mathbf{H} is defined as the minimum over all pseudoweights of nonzero pseudocodewords; this may be considered as a first-order measure of decoder error-correcting performance for LP or MP decoding. Typically, a lower minimum pseudoweight corresponds to a higher probability of decoding error. Another measure closely related to BSC pseudoweight is the max-fractional weight (pseudoweight). It serves as a lower bound on both AWGNC and BSC pseudoweights.

In order to minimise the decoding error probability under LP (or MP) decoding, one might want to select a matrix \mathbf{H} which maximises the minimum pseudoweight of the code for the given channel. However, generally it is not clear how this goal may be achieved. Adding redundant rows to the parity-check matrix introduces additional constraints on the so-called *fundamental cone*, and may thus increase the minimum pseudoweight. However, such additions increase the decoding complexity under MP decoding, especially since linear combinations of low-density rows may not yield a low-density result. On the other hand, there exist classes of codes for which sparse parity-check matrices exist with many redundant rows, e.g. [2].

For the AWGNC, BEC, BSC pseudoweights, and max-fractional weight, define $\rho_{\text{AWGNC}}(\mathcal{C})$, $\rho_{\text{BEC}}(\mathcal{C})$, $\rho_{\text{BSC}}(\mathcal{C})$, and $\rho_{\text{max-frac}}(\mathcal{C})$, respectively, to be the minimum number of rows in any parity-check matrix \mathbf{H} such that the minimum pseudoweight of \mathcal{C} with respect to this matrix is equal to the

code's minimum distance D . For the sake of simplicity, we sometimes use the notation $\rho(\mathcal{C})$ when the type of channel is clear from the context. The value $\rho(\mathcal{C})$ is called the (AWGNC, BEC, BSC, max-fractional) *pseudocodeword redundancy* (or pseudoredundancy) of \mathcal{C} . If for the code \mathcal{C} there exists no such matrix \mathbf{H} , we say that the pseudoredundancy is infinite.

The BEC pseudocodeword redundancy was studied in [3], where it was shown that for any binary linear code \mathcal{C} there exists a parity-check matrix \mathbf{H} such that the minimum pseudoweight with respect to this \mathbf{H} is equal to D , and therefore the BEC pseudocodeword redundancy is finite for all codes. The authors also presented some bounds on $\rho_{\text{BEC}}(\mathcal{C})$ for general linear codes, and for some specific families of codes.

In this work, we address the analogous problem for the AWGNC, BSC, and max-fractional pseudoweight. We show that for most codes there exists no \mathbf{H} such that the minimum pseudoweight (with respect to \mathbf{H}) is equal to D , and therefore the AWGNC, BSC, and max-fractional pseudocodeword redundancy (as defined above) is infinite for most codes. For some code families for which the pseudoredundancy is finite, we provide upper bounds on its value.

II. GENERAL SETTINGS

Let \mathcal{C} be a code of length $n \in \mathbb{N}$ over the binary field \mathbb{F}_2 , defined by

$$\mathcal{C} = \ker \mathbf{H} = \{\mathbf{c} \in \mathbb{F}_2^n : \mathbf{H}\mathbf{c}^T = \mathbf{0}^T\} \quad (1)$$

where \mathbf{H} is an $m \times n$ *parity-check matrix* of the code \mathcal{C} . Obviously, the code \mathcal{C} may admit more than one parity-check matrix, and all the codewords form a linear vector space of dimension $k \geq n - m$. We say that k is the *dimension* of the code \mathcal{C} . The *rate* of the code \mathcal{C} is defined as $R(\mathcal{C}) = k/n$ and is equal to the number of information bits per coded bit. We denote by D the minimum Hamming distance (also called the minimum distance) of \mathcal{C} . The code \mathcal{C} may then be referred to as an $[n, k, D]$ linear code over \mathbb{F}_2 .

Denote the set of column indices and the set of row indices of \mathbf{H} by $\mathcal{I} = \{1, 2, \dots, n\}$ and $\mathcal{J} = \{1, 2, \dots, m\}$, respectively. For $j \in \mathcal{J}$, we denote $\mathcal{I}_j \triangleq \{i \in \mathcal{I} : H_{j,i} \neq 0\}$, and for $i \in \mathcal{I}$, we denote $\mathcal{J}_i \triangleq \{j \in \mathcal{J} : H_{j,i} \neq 0\}$. The *fundamental cone* of \mathbf{H} , denoted $\mathcal{K}(\mathbf{H})$, is defined as the set of vectors $\mathbf{x} \in \mathbb{R}^n$ that satisfy

$$\forall j \in \mathcal{J}, \forall \ell \in \mathcal{I}_j : x_\ell \leq \sum_{i \in \mathcal{I}_j \setminus \{\ell\}} x_i, \quad (2)$$

$$\forall i \in \mathcal{I} : x_i \geq 0. \quad (3)$$

The vectors $\mathbf{x} \in \mathbb{R}^n$ satisfying (2) and (3) are called *pseudocodewords* of \mathcal{C} with respect to the parity-check matrix \mathbf{H} . Note that the fundamental cone $\mathcal{K}(\mathbf{H})$ depends on the parity-check matrix \mathbf{H} rather than on the code \mathcal{C} itself. At the same time, the fundamental cone is independent of the underlying communication channel.

The BEC, AWGNC, BSC pseudoweights, and max-fractional weight of a nonzero pseudocodeword $\mathbf{x} \in \mathcal{K}(\mathbf{H})$ were defined in [1] and [4] as follows:

$$\begin{aligned} w_{\text{BEC}}(\mathbf{x}) &\triangleq |\text{supp}(\mathbf{x})|, \\ w_{\text{AWGNC}}(\mathbf{x}) &\triangleq \frac{(\sum_{i \in \mathcal{I}} x_i)^2}{\sum_{i \in \mathcal{I}} x_i^2}. \end{aligned}$$

Let \mathbf{x}' be a vector in \mathbb{R}^n with the same components as \mathbf{x} but in non-increasing order. For $i-1 < \xi \leq i$, where $1 \leq i \leq n$, let $\phi(\xi) \triangleq x'_i$. Define $\Phi(\xi) \triangleq \int_0^\xi \phi(\xi') d\xi'$ and

$$w_{\text{BSC}}(\mathbf{x}) \triangleq 2 \Phi^{-1}(\Phi(n)/2).$$

Finally, the max-fractional weight of \mathbf{x} is defined as

$$w_{\text{max-frac}}(\mathbf{x}) \triangleq \frac{\sum_{i \in \mathcal{I}} x_i}{\max_{i \in \mathcal{I}} x_i}.$$

We define the BEC *minimum pseudoweight* of the code \mathcal{C} with respect to the parity-check matrix \mathbf{H} as

$$w_{\text{BEC}}^{\min}(\mathbf{H}) \triangleq \min_{\mathbf{x} \in \mathcal{K}(\mathbf{H}) \setminus \{\mathbf{0}\}} w_{\text{BEC}}(\mathbf{x}).$$

The quantities $w_{\text{AWGNC}}^{\min}(\mathbf{H})$, $w_{\text{BSC}}^{\min}(\mathbf{H})$ and $w_{\text{max-frac}}^{\min}(\mathbf{H})$ are defined similarly. Note that all four minimum pseudoweights are upper bounded by D , the code's minimum distance.

Then we define the BEC *pseudocodeword redundancy* of the code \mathcal{C} as

$$\rho_{\text{BEC}}(\mathcal{C}) \triangleq \inf\{\#\text{rows}(\mathbf{H}) \mid \ker \mathbf{H} = \mathcal{C}, w_{\text{BEC}}^{\min}(\mathbf{H}) = D\},$$

where $\inf \emptyset \triangleq \infty$, and similarly we define the pseudocodeword redundancies $\rho_{\text{AWGNC}}(\mathcal{C})$, $\rho_{\text{BSC}}(\mathcal{C})$ and $\rho_{\text{max-frac}}(\mathcal{C})$ for the AWGNC and BSC pseudoweights, and the max-fractional weight. We remark that all pseudocodeword redundancies satisfy $\rho(\mathcal{C}) \geq n - k$.

III. BASIC CONNECTIONS

The next lemma is taken from [4].

Lemma 3.1: Let \mathcal{C} be a binary linear code with the parity-check matrix \mathbf{H} . Then,

$$\begin{aligned} w_{\text{max-frac}}^{\min}(\mathbf{H}) &\leq w_{\text{AWGNC}}^{\min}(\mathbf{H}) \leq w_{\text{BEC}}^{\min}(\mathbf{H}), \\ w_{\text{max-frac}}^{\min}(\mathbf{H}) &\leq w_{\text{BSC}}^{\min}(\mathbf{H}) \leq w_{\text{BEC}}^{\min}(\mathbf{H}). \end{aligned}$$

The following theorem is a straightforward corollary to Lemma 3.1.

Theorem 3.2: Let \mathcal{C} be a binary linear code. Then,

$$\begin{aligned} \rho_{\text{max-frac}}(\mathcal{C}) &\geq \rho_{\text{AWGNC}}(\mathcal{C}) \geq \rho_{\text{BEC}}(\mathcal{C}), \\ \rho_{\text{max-frac}}(\mathcal{C}) &\geq \rho_{\text{BSC}}(\mathcal{C}) \geq \rho_{\text{BEC}}(\mathcal{C}). \end{aligned}$$

We note that for *geometrically perfect* codes, a class of codes defined and characterised in [5], all four pseudocodeword redundancies are finite.

IV. PSEUDOREDUNDANCY OF RANDOM CODES

We begin with the following lemma.

Lemma 4.1: For the binary linear code \mathcal{C} of length n , let d be the minimum distance of the dual code. Then, the minimum AWGNC pseudoweight of \mathcal{C} (with respect to any parity-check matrix \mathbf{H}) satisfies

$$w_{\text{AWGNC}}^{\min} \leq \frac{(n + d - 2)^2}{(d - 1)^2 + (n - 1)}. \quad (4)$$

Proof: Consider the pseudocodeword $\mathbf{x} = (x_1, x_2, \dots, x_n) \triangleq (d - 1, 1, \dots, 1)$. Since d is the minimum distance of the dual code, every row in \mathbf{H} has weight at least d . Therefore, all inequalities (2) and (3) are satisfied for this \mathbf{x} , and so it is indeed a legal pseudocodeword. Finally, observe that the AWGNC pseudoweight of \mathbf{x} is given by the right-hand side of (4). ■

Next, we take a random binary linear code \mathcal{C} of a fixed rate R and arbitrary length n (for $n \rightarrow \infty$). It is well known that the relative minimum distance $\delta = D/n$ of \mathcal{C} attains, with probability approaching 1 as $n \rightarrow \infty$, for any fixed small $\epsilon > 0$, the Gilbert-Varshamov bound

$$\delta \geq H_2^{-1}(1 - R) - \epsilon,$$

where $H_2^{-1}(\cdot)$ is the inverse of the binary entropy function $H_2(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$ (see [6, Theorems 4.4, 4.5, and 4.10] for details).

The dual code \mathcal{C}^\perp of \mathcal{C} can be viewed as a random code also, and so with high probability the rate $R^\perp = 1 - R$ and the relative minimum distance $\delta^\perp = d/n$ of the dual code attain the Gilbert-Varshamov bound

$$\delta^\perp \geq \mu \triangleq H_2^{-1}(1 - R^\perp) - \epsilon = H_2^{-1}(R) - \epsilon,$$

Note that (4) may be written in terms of the relative minimum distance δ^\perp of the dual code as follows:

$$w_{\text{AWGNC}}^{\min} \leq \frac{(1 + \delta^\perp - 2/n)^2}{(\delta^\perp - 1/n)^2 + (1/n - 1/n^2)}. \quad (5)$$

For large n , the minimum pseudoweight of the code \mathcal{C}^\perp is bounded from above by the constant $(1 + 1/\delta^\perp)^2 + \epsilon' \leq (1 + 1/\mu)^2 + \epsilon'$ for some small $\epsilon' > 0$ — this bound does not depend on n . On the other hand, \mathcal{C} is a random code and so its minimum distance satisfies the Gilbert-Varshamov bound, namely

$$D \geq (H_2^{-1}(1 - R) - \epsilon) \cdot n,$$

which increases linearly with n for a fixed R .

We obtain that for any \mathbf{H} , there is a gap between the minimum pseudoweight and the minimum distance of a random code \mathcal{C} . Therefore, we have the following corollary.

Corollary 4.2: Let $0 < R < 1$ be fixed. For n large enough, for a random binary linear code \mathcal{C} of length n and rate R , there is a gap between the minimum AWGNC pseudoweight (with respect to any parity-check matrix) and the minimum distance. Therefore, the AWGNC pseudoredundancy is infinite for most codes.

The following lemma is a counterpart of Lemma 4.1 for the BSC.

Lemma 4.3: Let \mathcal{C} be a binary linear code of length n , and let d be the minimum distance of the dual code. Then, the minimum BSC pseudoweight of \mathcal{C} (with respect to any parity-check matrix \mathbf{H}) satisfies

$$w_{\text{BSC}}^{\min} \leq 2\lceil n/d \rceil.$$

Proof: Consider the pseudocodeword

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \triangleq (\underbrace{d-1, d-1, \dots, d-1}_{\tau}, \underbrace{1, 1, \dots, 1}_{n-\tau}),$$

for some positive integer τ . This \mathbf{x} is then a legal pseudocodeword; since d is the minimum distance of the dual code, every row in \mathbf{H} has a weight of at least d , and so, all inequalities (2) and (3) are satisfied by this \mathbf{x} .

If $\tau(d-1) \geq n - \tau$ then by the definition of the BSC pseudoweight $w_{\text{BSC}}(\mathbf{x}) \leq 2\tau$. This condition is equivalent to $\tau d \geq n$. Therefore, we set $\tau = \lceil n/d \rceil$. For the corresponding \mathbf{x} , the pseudoweight is less or equal to $2\tau = 2\lceil n/d \rceil$. ■

Similarly to the AWGNC case, let \mathcal{C} be a random binary code of length n and a fixed rate R . The parameters R^\perp and δ^\perp of its dual code \mathcal{C}^\perp attain with high probability the Gilbert-Varshamov bound $\delta^\perp \geq \mu$.

From Lemma 4.3, for all n , the pseudoweight of the code \mathcal{C}^\perp is bounded from above by

$$2\lceil n/d \rceil < 2/\delta^\perp + 2 \leq 2/\mu + 2,$$

which is a constant. On the other hand, \mathcal{C} is a random code and its minimum distance also satisfies the Gilbert-Varshamov bound, so it increases linearly with n . It follows that for any \mathbf{H} , there is a gap between the minimum BSC pseudoweight and the minimum distance of a random code \mathcal{C} .

Corollary 4.4: Let $0 < R < 1$ be fixed. For n large enough, for a random binary linear code \mathcal{C} of length n and rate R , there is a gap between the minimum BSC pseudoweight (with respect to any parity-check matrix) and the minimum distance. Therefore, the BSC pseudoredundancy is infinite for most codes.

The last corollary disproves the conjecture in [7] that the BSC pseudoredundancy is finite for all binary linear codes.

Example 4.1: Consider the [23,12] Golay code having minimum distance $D = 7$. The minimum distance of its dual code is $d = 8$. We can take a pseudocodeword \mathbf{x} as in the proof of Lemma 4.3 with $\tau = 3$. We have $w_{\text{BSC}}(\mathbf{x}) \leq 2\tau = 6$, thus obtaining that the minimum distance is not equal to the minimum pseudoweight.

Similarly, for the [24,12] extended Golay code we have $D = d = 8$, and by taking $\tau = 3$ we obtain $w_{\text{BSC}}(\mathbf{x}) \leq 2\tau = 6$.

Note however that the presented techniques do not answer the question of whether these Golay codes have finite AWGNC pseudoredundancy.

V. CODES BASED ON DESIGNS

Definition 5.1: A partial (w_c, λ) design is a block design consisting of an n -element set \mathcal{V} (whose elements are called *points*) and a collection of m subsets of \mathcal{V} (called *blocks*) such that every point is contained in exactly w_c blocks and every 2-element subset of \mathcal{V} is contained in at most λ blocks. The *incidence matrix* of a design is an $m \times n$ matrix \mathbf{H} whose rows correspond to the blocks and whose columns correspond to the points, and satisfies $H_{j,i} = 1$ if block j contains point i , and $H_{j,i} = 0$ otherwise.

If each block contains the same number w_r of points and every 2-element subset of \mathcal{V} is contained in exactly λ blocks, the design is said to be an (n, w_r, λ) *balanced incomplete block design* (BIBD).

Note that for a BIBD we have $nw_c = mw_r$ and also

$$w_c(w_r - 1) = \lambda(n - 1), \quad (6)$$

so all other parameters may be deduced from (n, w_r, λ) ; in particular, $w_c = \frac{n-1}{w_r-1}\lambda$. Note that [9] and [10] consider parity-check matrices based on BIBDs; these matrices are the transpose of the incidence matrices defined here.

We have the following general result for codes based on partial (w_c, λ) designs.

Theorem 5.1: Let \mathcal{C} be a code with parity-check matrix \mathbf{H} , such that a subset of the rows of \mathbf{H} forms the incidence matrix for a partial (w_c, λ) design. Then the minimum max-fractional weight of \mathcal{C} with respect to \mathbf{H} is lower bounded by

$$w_{\text{max-fac}}^{\min} \geq 1 + \frac{w_c}{\lambda}. \quad (7)$$

For the case of an (n, w_r, λ) BIBD, the lower bound in (7) may also be written as $1 + \frac{n-1}{w_r-1}$; the alternative form follows trivially from (6).

Proof: Consider the subset of the rows of \mathbf{H} which forms the incidence matrix for a partial (w_c, λ) design. Let \mathbf{x} be a nonzero pseudocodeword and let x_ℓ be a maximal coordinate of \mathbf{x} ($\ell \in \mathcal{I}$). For all $j \in \mathcal{J}$ such that $\ell \in \mathcal{I}_j$, sum inequalities (2). We have

$$w_c \cdot x_\ell \leq \lambda \cdot \sum_{i \in \mathcal{I} \setminus \{\ell\}} x_i,$$

or

$$\left(1 + \frac{w_c}{\lambda}\right) x_\ell \leq \sum_{i \in \mathcal{I}} x_i. \quad (8)$$

The result now easily follows from the definition of $w_{\text{max-fac}}^{\min}$. ■

Theorem 5.2: Let \mathcal{C} be a code with parity-check matrix \mathbf{H} , such that a subset of the rows of \mathbf{H} forms the incidence matrix for a partial (w_c, λ) design. Then,

$$w_{\text{AWGNC}}^{\min} \geq 1 + \frac{w_c}{\lambda},$$

$$w_{\text{BSC}}^{\min} \geq 1 + \frac{w_c}{\lambda}.$$

The proof follows from Lemma 3.1 and Theorem 5.1.

Another tool for proving lower bounds of the minimum AWGNC pseudoweight is provided by the following eigenvalue-based bound from [11].

Proposition 5.3: The minimum AWGNC pseudoweight for a (w_c, w_r) -regular parity-check matrix \mathbf{H} whose corresponding Tanner graph is connected is bounded below by

$$w_{\text{AWGNC}}^{\min} \geq n \cdot \frac{2w_c - \mu_2}{\mu_1 - \mu_2}, \quad (9)$$

where μ_1 and μ_2 denote the largest and second largest eigenvalue (respectively) of the matrix $\mathbf{L} \triangleq \mathbf{H}^T \mathbf{H}$, considered as a matrix over the real numbers.

In the case where \mathbf{H} is equal to the incidence matrix for an (n, w_r, λ) BIBD, it is easy to check that the bound of Proposition 5.3 becomes

$$w_{\text{AWGNC}}^{\min} \geq 1 + \frac{w_c}{\lambda}.$$

We conclude that in this case the bound of Proposition 5.3 coincides with that of Theorem 5.2 (for the case of the AWGNC only).

Note that the pseudoweight bounds of [12] for the EG(2, q) and PG(2, q) codes for $q = 2^s \geq 2$ follow from Theorem 5.2. We next apply the bounds of Theorems 5.1 and 5.2 to some other examples of codes derived from designs.

Proposition 5.4: For $m \geq 2$, the $[2^m - 1, 2^m - 1 - m, 3]$ Hamming code has AWGNC, BSC, and max-fractional pseudocodeword redundancies $\rho(\mathcal{C}) \leq 2^m - 1$.

Proof: For $m \geq 2$, consider the binary parity-check matrix \mathbf{H} whose rows are exactly the nonzero codewords of the dual code \mathcal{C}^\perp , in this case the $[2^m - 1, m, 2^{m-1}]$ simplex code. This \mathbf{H} is the incidence matrix for a BIBD with parameters $(n, w_r, \lambda) = (2^m - 1, 2^{m-1}, 2^{m-2})$. Theorem 5.1 gives $w_{\text{max-fac}}(\mathbf{x}) \geq 3$, leading to $\rho_{\text{max-fac}}(\mathcal{C}) \leq 2^m - 1$. The result for AWGNC and BSC follows by applying Theorem 3.2. ■

In the next example, we consider simplex codes. Straightforward application of the previous reasoning does not lead to the desired result. However, more careful selection of the matrix \mathbf{H} , as described below, leads to a new bound on the pseudoredundancy.

Proposition 5.5: For $m \geq 2$, the $[2^m - 1, m, 2^{m-1}]$ simplex code has AWGNC, BSC, and max-fractional pseudocodeword redundancies

$$\rho(\mathcal{C}) \leq \frac{(2^m - 1)(2^{m-1} - 1)}{3}.$$

Proof: For $m \geq 2$, consider the binary parity-check matrix \mathbf{H} whose rows are exactly the codewords of the dual code \mathcal{C}^\perp (in this case the $[2^m - 1, 2^m - 1 - m, 3]$ Hamming code) with Hamming weight equal to 3. This \mathbf{H} is the incidence matrix for a BIBD with parameters $(n, w_r, \lambda) = (2^m - 1, 3, 1)$. Theorem 5.1 gives $w_{\text{max-fac}}^{\min} \geq 2^{m-1}$.

Note that the number of codewords of weight 3 in the $[2^m - 1, 2^m - 1 - m, 3]$ Hamming code is $(2^m - 1)(2^{m-1} - 1)/3$. This is due to the fact that there exists a 3 : 1 mapping from all vectors of length $2^m - 1$ and weight 2 onto the codewords of weight 3.

Next, we justify the claim that \mathbf{H} is the parity-check matrix of \mathcal{C} . A theorem of Simonis [13] states that if there

exists a linear $[n, k, D]$ code then there also exists a linear $[n, k, D]$ code whose codewords are spanned by the codewords of weight D . Since the Hamming code is unique for the parameters $[2^m - 1, 2^m - 1 - m, 3]$, this implies that the Hamming code itself is spanned by the codewords of weight 3, so the row space of \mathbf{H} equals \mathcal{C} .

The result for AWGNC and BSC follows again by applying Theorem 3.2. ■

We remark that the bounds of Propositions 5.4 and 5.5 are sharp at least for the case $m = 3$ and the max-fractional weight, see Section VI-B.

The following proposition proves that the AWGNC, BSC, and max-fractional pseudocodeword redundancies are finite for all codes \mathcal{C} with minimum distance at most 3.

Proposition 5.6: Let \mathcal{C} be a $[n, k, D]$ code with $D \leq 3$. Then $\rho_{\text{max-fac}}(\mathcal{C})$ is finite. Moreover, we have $\rho_{\text{max-fac}}(\mathcal{C}) = n - k$ in the case $D \leq 2$.

Proof: First assume that $D \leq 2$. Let \mathbf{H} be any parity-check matrix for the code \mathcal{C} , let \mathbf{x} be a nonzero pseudocodeword, and assume that x_ℓ is a maximal entry in \mathbf{x} (for some $\ell \in \mathcal{I}$). We always have $\sum_{i \in \mathcal{I}} x_i \geq x_\ell$ and hence $w_{\text{max-fac}}(\mathbf{x}) \geq 1$.

Therefore, we may assume $D = 2$. Note that for such a code, \mathbf{H} has no zero column and thus we may write by (2)

$$x_\ell \leq \sum_{i \in \mathcal{I} \setminus \{\ell\}} x_i, \quad \text{or} \quad 2x_\ell \leq \sum_{i \in \mathcal{I}} x_i.$$

From the definition of max-fractional weight, we obtain that $w_{\text{max-fac}}(\mathbf{x}) \geq 2$. Choosing a parity-check matrix for \mathcal{C} with $n - k$ rows we have that $\rho_{\text{max-fac}}(\mathcal{C}) = n - k$. From Theorem 3.2, $\rho_{\text{AWGNC}}(\mathcal{C}) = n - k$ and $\rho_{\text{BSC}}(\mathcal{C}) = n - k$.

Next, consider a code with minimum distance $D = 3$. Denote by \mathbf{H} the parity-check matrix whose rows consist of all codewords of the dual code of \mathcal{C} . Note that for a code of minimum distance D , a parity-check matrix \mathbf{H} consisting of all rows of the dual code \mathcal{C}^\perp is an orthogonal array of strength $D - 1$. In the present case $D = 3$, and this implies that in any pair of columns of \mathbf{H} , all length-2 binary vectors occur with equal multiplicities (c.f. [14, p. 139]). Thus the matrix \mathbf{H} is an incidence matrix for a partial block design with parameters $(w_c, \lambda) = (2^{r-1}, 2^{r-2})$, where $r = n - k$. Therefore for this matrix \mathbf{H} the code has minimum (AWGNC, BSC, or max-fractional) pseudoweight at least $1 + w_c/\lambda = 3$, and it follows that the pseudocodeword redundancy is finite for any code with $D = 3$. ■

VI. SOME EXPERIMENTAL RESULTS

A. Cyclic Codes Meeting the Eigenvalue Bound

We consider cyclic codes of length n with full circulant parity-check matrix \mathbf{H} . Thus $\mathbf{H} = (H_{j,i})_{i,j \in \mathcal{I}}$ is a square matrix with entries $H_{j,i} = c_{j-i}$ for some vector \mathbf{c} of length n , where all the indices are modulo n . This $n \times n$ matrix is then w -regular (i.e. (w, w) -regular), where $w = \sum_{i \in \mathcal{I}} c_i$, so we may use the eigenvalue-based lower bound in Proposition 5.3 to examine the AWGNC pseudocodeword redundancy.

TABLE I
CYCLIC CODES UP TO LENGTH 250 WITH $D \geq 3$ MEETING THE
EIGENVALUE BOUND

parameters	w -regular	comments
$[n, 1, n]$	2	repetition code, $n = 3 \dots 250$
$[n, n - m, 3]$	2^{m-1}	Hamming code, $n = 2^m - 1$, $m = 3 \dots 7$
$[7, 3, 4]$	3	dual of the $[7, 4, 3]$ Hamming code
$[15, 7, 5]$	4	Euclidean geometry code EG(2,4)
$[21, 11, 6]$	5	projective geometry code PG(2,4)
$[63, 37, 9]$	8	Euclidean geometry code EG(2,8)
$[73, 45, 10]$	9	projective geometry code PG(2,8)

For the largest eigenvalue of the matrix $\mathbf{L} = \mathbf{H}^T \mathbf{H}$ we have $\mu_1 = w^2$, since every row weight of \mathbf{L} equals $\sum_{i,j \in \mathcal{I}} c_i c_j = w^2$. Consequently, the eigenvalue bound is

$$w_{\text{AWGNC}}^{\min} \geq n \frac{2w - \mu_2}{w^2 - \mu_2},$$

where μ_2 is the second largest eigenvalue of \mathbf{L} .

We carried out an exhaustive search on all cyclic codes \mathcal{C} up to length $n \leq 250$ and computed the eigenvalue bound in all cases where the Tanner graph of the full circulant parity-check matrix is connected.¹ Table I gives a complete list of all cases in which the eigenvalue bound equals the minimum Hamming distance D and $D \geq 3$. In particular, the AWGNC pseudoweight equals the minimum Hamming distance in these cases as well and thus we have for the pseudocodeword redundancy $\rho_{\text{AWGNC}}(\mathcal{C}) \leq n$.

B. The Pseudocodeword Redundancy for Codes of Small Length

Let \mathcal{C} be a binary linear code with parameters $[n, k, D]$ and let $r = n - k$. Two parity-check matrices \mathbf{H} and \mathbf{H}' of \mathcal{C} are called *equivalent* if \mathbf{H} can be transformed into \mathbf{H}' by a sequence of row and column permutations. In this case, $w_{\min}(\mathbf{H}) = w_{\min}(\mathbf{H}')$.

We computed the AWGNC, BSC, and max-fractional pseudocodeword redundancies for all codes up to length 9. Note that for this also all possible parity-check matrices (up to equivalence) had to be examined.² The main observations are the following:

- If $D \geq 3$ then for *every* parity-check matrix \mathbf{H} we have $w_{\text{AWGNC}}^{\min}(\mathbf{H}) \geq 3$. This is not true for the BSC.
- If $k = 2$, then $\rho_{\text{AWGNC}}(\mathcal{C}) = \rho_{\text{BSC}}(\mathcal{C}) = \rho_{\text{max-fractional}}(\mathcal{C}) = r$.
- For the $[7, 4, 3]$ Hamming code \mathcal{C} we have $\rho_{\text{AWGNC}}(\mathcal{C}) = r = 3$, $\rho_{\text{BSC}}(\mathcal{C}) = 4$, and $\rho_{\text{max-fractional}}(\mathcal{C}) = 7$.
- For the $[7, 3, 4]$ simplex code \mathcal{C} we have $\rho_{\text{AWGNC}}(\mathcal{C}) = r = 4$, $\rho_{\text{BSC}}(\mathcal{C}) = 5$, and $\rho_{\text{max-fractional}}(\mathcal{C}) = 7$. There is (up to equivalence) only one parity-check matrix \mathbf{H} with

$w_{\text{AWGNC}}^{\min}(\mathbf{H}) = 4$, namely

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

It is the only parity-check matrix with constant row weight 3.

- For the $[8, 4, 4]$ extended Hamming code \mathcal{C} we have $\rho_{\text{AWGNC}}(\mathcal{C}) = 5$, $\rho_{\text{BSC}}(\mathcal{C}) = 6$, and $\rho_{\text{max-fractional}}(\mathcal{C}) = \infty$. This code \mathcal{C} is the shortest one such that $\rho_{\text{AWGNC}}(\mathcal{C}) > r$, and also the shortest one such that $\rho_{\text{max-fractional}}(\mathcal{C}) = \infty$.

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¹This computation was done by a self-written C program.

²The enumeration of codes and parity-check matrices was done by self-written C programs. The minimum pseudoweight for the various parity-check matrices was computed by using Maple 12 and the Convex package by Matthias Franz, available at <http://www-fourier.ujf-grenoble.fr/~franz/convex/>.